Hamiltonian formulation for the motion of vortices in the presence of a free surface for ideal flow

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In this paper, the work of Zakharov [J. Appl. Mech. Tech. Phys., Engl. Transl. **2**, 190 (1968)] giving the canonical-Hamiltonian formulation for irrotational ideal flow with a free surface is generalized to the case in which the fluid interior contains isolated vortices. In particular, for two-dimensional flow the case of point vortices in the fluid interior is considered and for three-dimensional flow the case of vortex filaments in the interior is considered. Canonical variables are obtained explicitly for each of these cases. In the idealization of infinitely thin filaments, one has the usual problem of an infinite normal velocity being induced on a curved filament, so that a finite thickness must be given to each filament. This procedure is handled naturally by the Hamiltonian formulation given.

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I. INTRODUCTION

In this paper we develop a canonical-Hamiltonian theory of fluid flows involving a free surface and either point vortices in two dimensions, or vortex filaments in three dimensions. Both of these vortical flows have been well studied in fluids with fixed boundaries, and have provided considerable insight into flows with nonsingular vorticity. Although the equations of motion are well known, the free surface adds considerable complexity and approximations have to be made in order to study the problem.

Hamiltonian methods often have advantages when approximations must be made. There are several reasons for this, one of which is that conservation laws are easier to respect. Another important advantage is the fact that there is an underlying variational principle which is extremely useful for problems in which there are space- and/or time-scale separations, as discussed extensively by Whitham [1]. We also mention that it is easier to keep track of orders of perturbation theory in a Hamiltonian framework. Previous calculations using Hamiltonian methods for surface waves in the absence of vorticity have demonstrated the usefulness of the Hamiltonian formulation [2–10]. It is with the expectation that similar benefits will arise for simple vorticity configurations that we extend the Hamiltonian formalism for surface waves to include point vortices and vortex filaments in the interior. Additionally, the Hamiltonian equations of motion that we obtain for vortex filaments are of a mixed Lagrangian-Eulerian type which is different from the usual formulations. It is at present unknown whether this formulation would offer an advantage in numerical computations, but there is at least a possibility of improvement.

This paper will be devoted to the development of the formalism. In subsequent publications we will calculate the interaction between point vortices and surface waves for some typical examples. We will also consider $3d$ vortex filaments in the local induction approximation [11], interacting with surface waves. This provides an interesting model of the trailing vortices in a ship wake.

In the Introduction we review the Hamiltonian forms of the free surface motion and that of point-vortex motion separately. In the next section we derive the canonical-Hamiltonian form of the coupled equations for point vortices and a free surface in two dimensions. Although the two-dimensional point vortex has been studied in Hamiltonian form for a long time, there has been no comparable formulation of vortex filaments in a canonical-Hamiltonian theory. In Sec. III we derive the canonical-Hamiltonian form for vortex filaments; this provides a description of vortex filament motion different from the usual ones. It is of a mixed Lagrangian-Eulerian form and potentially could have useful computational applications. We then combine filament motion with free surface motion. Next we show how a Hamiltonian form of the local induction approximation [11] can be derived from our formulation, and more generally how the Hamiltonian can be modified to remove the infinite self-induced velocity of a vortex filament. In Sec. IV we present Hamilton's variational principle. The three appendixes contain most of the lengthy algebraic calculations.

A. Review of Hamiltonian formulations for point-vortex and free surface motion

It has been known since the last century that the motion of point vortices in a two-dimensional ideal fluid, either unbounded or with rigid boundaries, is governed by Hamilton's canonical equations of motion, where the Hamiltonian is given by the "excess energy" of the vortex system and the canonical variables are the x and y positions of the vortices [12]. More explicitly, if the fluid is unbounded and we have N vortices with the αth vortex located at $\mathbf{X}_\alpha = (X_\alpha, Y_\alpha)$, then the vortex motion is governed by

$$\frac{dX_\alpha}{dt} = \{X_\alpha, H\}, \quad \frac{dY_\alpha}{dt} = \{Y_\alpha, H_p\} \tag{1}$$

with the Hamiltonian H_p given by

$$H_p = -\frac{1}{4\pi}\sum_{\alpha}\sum_{\beta\,(\neq\alpha)} \Gamma_\alpha\Gamma_\beta \ln|\mathbf{X}_\alpha - \mathbf{X}_\beta|, \tag{2}$$

where Γ_α is the circulation of the αth vortex. The subscript p is for point vortex, and is used to distinguish this Hamiltonian from others which we shall encounter later. The Poisson bracket is an "almost" canonical one given by

$$\{X_\alpha, Y_\beta\} = \frac{\delta_{\alpha\beta}}{\Gamma_\alpha}. \tag{3}$$

The coordinates can be made canonical by using $\sqrt{\pm\Gamma_\alpha}\mathbf{X}_\alpha$ instead of $\mathbf{X}_\alpha$, depending on the sign of Γ_α. There is no advantage to doing this, however, and we shall use the vortex positions $\mathbf{X}_\alpha$ in what follows.

The vortex motion problem given above gives rise to very rich and complicated behavior (for a review of which the reader may consult Aref [13]). However, here we are not directly concerned with this problem. We will instead describe briefly the well known Hamiltonian formulation for another old problem in ideal flow. Namely the motion of an irrotational fluid bounded by a free surface, under the influence of gravity. Our ultimate goal is to combine the vortex and free surface systems. The free surface problem in Hamiltonian form is discussed by a number of authors, among whom are Zakharov [2], Broer [3], Watson and West [4], Miles [5], Milder [6, 7], West *et. al.* [8], Henyey *et. al.* [9], and Creamer *et. al.* [10]. Here we consider the two-dimensional problem to set the stage, though the three-dimensional generalization is straightforward and will be discussed later, after we discuss vortex filaments.

Let the position of the free surface be given by $y = \zeta(x,t)$ (that the free surface may be described in this fashion is, of course, an assumption, later we will briefly discuss more general situations, such as breaking waves), then the irrotational velocity field $\mathbf{u}(\mathbf{x},t) = [u(\mathbf{x},t), v(\mathbf{x},t)]$ of the fluid located in the interior $[y \le \zeta(x,t)]$ can be described in terms of a velocity potential $\Phi(\mathbf{x},t)$,

$$\mathbf{u}(\mathbf{x},t) = (\partial_x, \partial_y)\Phi(\mathbf{x},t) = \boldsymbol{\nabla}\Phi(\mathbf{x},t). \tag{4}$$

Since the fluid is incompressible, the velocity potential satisfies the Laplace equation

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \nabla^2\Phi = 0. \tag{5}$$

Assuming that $\mathbf{u} \to 0$ as $y \to -\infty$ (so that we have no motion as $y \to -\infty$) the solution to the Laplace equation for Φ is completely determined by specifying the boundary conditions at the free surface. [However, it should be noted that it is straightforward to extend this analysis, and all of what follows, to the case when one has a rigid and (in general) curved bottom and side boundaries.] Denoting the boundary value of Φ by $\phi(x,t) = \Phi(x,\zeta(x,t),t)$, we see that specifying $\phi(x,t)$ and $\zeta(x,t)$ determines the fluid motion at time t. We therefore need to obtain evolution equations for ϕ and ζ. These are obtained by applying the dynamic and kinematic boundary conditions at the free surface. That is the evolution equation for ϕ is obtained by evaluating the Bernoulli equation on the free surface and using the fact the pressure on the free surface is a constant (which we set to zero), while the evolution equation for ζ is obtained simply by using the fact that the free surface velocity is the fluid velocity evaluated on the free surface. These lead to the following equations of motion:

$$\begin{aligned} \phi_t &= u_2^s(u_2^s - u_1^s\zeta_x) - \frac{|\mathbf{u}^s|^2}{2} - g\zeta, \\ \zeta_t &= u_2^s - u_1^s\zeta_x, \end{aligned} \tag{6}$$

where the $\mathbf{x}$ and t subscripts denote partial differentiation and $\mathbf{u}^s$ is the fluid velocity evaluated on the free surface,

$$\mathbf{u}^s(x,t) = (u_1^s(x,t), u_2^s(x,t)) = \mathbf{u}(x,\zeta(x,t),t) = (\nabla\Phi)(x,\zeta(x,t),t), \tag{7}$$

and g is the acceleration due to gravity. Thus even though the equations of motion (6) involve only quantities evaluated on the free surface, evaluation of the right-hand sides of the evolution equations for ϕ and ζ actually does require the solutions of the Laplace equation for Φ. Therein lies the main difficulty in calculating numerical solutions of Eqs. (6).

About 25 years ago, Zakharov [2] discovered that the evolution equations (6) actually describe a canonical-Hamiltonian system with ζ and ϕ being the canonical coordinate and momentum, respectively, and the Hamiltonian being the energy of the system. That is Eqs. (6) can be written

$$\zeta_t = \{\zeta, H\}, \quad \phi_t = \{\phi, H\}, \tag{8}$$

with

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\zeta} \frac{|\boldsymbol{\nabla}\Phi|^2}{2} dy\,dx + \int_{-\infty}^{\infty} \frac{g\zeta^2}{2} dx, \tag{9}$$

where the first integral (which is over the bulk of the fluid) is the kinetic energy, and the second integral is the gravitational potential energy. The Poisson bracket is the canonical bracket given by

$$\{\zeta(x), \phi(x')\} = \delta(x - x'). \tag{10}$$

We now make a couple of remarks. First, the kinetic energy term in the Hamiltonian can also be written as an integral over the surface by an application of Green's theorem. However this "bulk form" of the kinetic energy has the advantage that its physical meaning is clear and also it brings out the fact that the Hamiltonian is a rather complicated functional of the canonical variables, that is it requires the solution of the Laplace equation for Φ, with the canonical variables coming in through the boundary condition $\Phi|_{y=\zeta} = \phi$. The demonstration that Hamilton's equations (8) are equivalent to the equations of motion (6) is therefore not an altogether trivial exercise. However, we leave this calculation to the next section where we generalize the problem to the case where the interior flow is not altogether irrotational but includes isolated point vortices. The irrotational problem discussed by Zakharov then follows as a special case.

II. FREE SURFACE MOTION IN TWO DIMENSIONS WITH POINT VORTICES IN THE INTERIOR

Now we imagine that as above, we have unbounded fluid motion, except for a free surface at $y = \zeta(x,t)$, but instead of irrotational motion in the interior we have N point vortices located at (X_α, Y_α), $\alpha = 1,\ldots,N$. At any later time, the fluid configuration will again consist of a free surface and N point vortices. Therefore, we expect that the above fluid configuration will again be a canonical-Hamiltonian system. As a good first guess we expect that the combination of the canonical coordinates for the vortex and free surface problems discussed above will constitute the canonical coordinates for the combined problem of interest here. This guess turns out to be basically correct, except that the status of one of the proposed canonical variables, namely the boundary value of the velocity potential on the free surface, $\phi(x,t)$, is now ambiguous. This is so because there are two candidates for velocity potential in this problem, namely the velocity potential in the absence of the vortices, and the one with the vortices included. Now the latter quantity is certainly multiple valued, since even though the flow is potential everywhere in the interior (except at the location of the vortices), the finite circulation around each vortex means that the velocity potential must change by $2\pi\Gamma$ as one makes a closed loop around the vortex with circulation Γ. However it turns out that it is the surface value of this "total" velocity potential which is the canonical momentum conjugate to surface elevation $\zeta(x,t)$; a fact which we verify explicitly below. The multiple valuedness of the potential can be removed by drawing an imaginary line from the vortex position to a boundary which we take to be at $y = -\infty$. With this prescription the potential is single-valued everywhere and there is no jump in the potential along the free surface. Green's theorem has to be appropriately modified to take into account the new imaginary boundary.

A. The equations of motion

We now write down the equations of motion for our free surface plus point-vortex system. As before, let ζ be the elevation of the free surface, $\mathbf{u}$ the fluid velocity in the bulk, $\mathbf{u}^s$ the fluid velocity evaluated at the free surface, and (X_α, Y_α) the position of the αth vortex with circulation Γ_α. We now establish the following notation:

$$\mathbf{U}^\alpha(\mathbf{x},t) = \frac{\Gamma_\alpha}{2\pi}\frac{(-(y-Y_\alpha), x-X_\alpha)}{|\mathbf{x}-\mathbf{X}^\alpha|^2}, \tag{11}$$

$$\mathbf{U}(\mathbf{x},t) = \sum_\alpha \mathbf{U}^\alpha(\mathbf{x},t), \tag{12}$$

$$\mathbf{v}^\alpha(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{U}^\alpha(\mathbf{x},t), \tag{13}$$

$$\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{U}(\mathbf{x},t). \tag{14}$$

Thus $\mathbf{U}^\alpha$ is the velocity field induced by a point vortex located at $\mathbf{X}^\alpha$ with strength Γ_α *in an unbounded fluid.* $\mathbf{U}$ is the sum of such velocities for all of our vortices, while $\mathbf{v}^\alpha$ and $\mathbf{v}$ are velocity fields obtained by removing the contributions of a single vortex and all vortices, respectively, from the total velocity field $\mathbf{u}$. We shall use a superscript s for any of these velocity fields evaluated on the free surface, for example, $\mathbf{v}^s(x,t) = \mathbf{v}(x,\zeta,t)$. Next, noting that each of these velocity fields is irrotational away from the vortices, we define various velocity potentials (which may be multiple-valued), through the following relations:

$$\mathbf{u} = \nabla\Phi, \tag{15}$$

$$\mathbf{U}^\alpha = \nabla\Phi_p^\alpha, \tag{16}$$

$$\mathbf{U} = \nabla\Phi_p, \tag{17}$$

$$\mathbf{v} = \nabla\Lambda. \tag{18}$$

The subscript p on Φ_p^α and Φ_p is for point vortex. For these quantities we have the explicit formulas

$$\Phi_p^\alpha = -\frac{\Gamma_\alpha}{2\pi}\arctan\left(\frac{x-X_\alpha}{y-Y_\alpha}\right), \tag{19}$$

$$\Phi_p = -\sum_\alpha \frac{\Gamma_\alpha}{2\pi}\arctan\left(\frac{x-X_\alpha}{y-Y_\alpha}\right). \tag{20}$$

Thus, in general, Φ_p^α, Φ_p, and Φ are multiple-valued due to the presence of the vortices, but Λ is always single-valued since the singularities due to the vortices are explicitly excluded from it. However, it can be seen now that as claimed previously, on the free surface $y = \zeta(x,t)$, Φ_p and Φ are smooth and single-valued, so long as the vortices are below the free surface.

Finally, we will denote the surface values of these velocity potentials by lower case letters, thus ϕ, ϕ_p, ϕ_p^α, and λ are Φ, Φ_p, Φ_p^α, and Λ evaluated at $(x,y) = (x,\zeta(x,t))$. With this notation we can now immediately write down the equations of motion for our candidate canonical set $(\zeta, \phi, X_\alpha, Y_\alpha)$

$$\begin{aligned}\frac{d\mathbf{X}^\alpha}{dt} &= \mathbf{v}^\alpha(\mathbf{X}^\alpha),\\ \phi_t &= u_2^s(u_2^s - u_1^s\zeta_x) - \frac{|\mathbf{u}^s|^2}{2} - g\zeta,\\ \zeta_t &= u_2^s - u_1^s\zeta_x.\end{aligned} \tag{21}$$

The equations of motion for the vortices are simply the statement that the vortices move with the fluid. Near each vortex, the dominant velocity is the rotational one induced by the vortex itself, however this velocity (which is singular at the position of the vortex) does not influence the motion of the vortex and must be subtracted from the total velocity. This is why we have $\mathbf{v}^\alpha$ rather than $\mathbf{u}$ on the right-hand side of the first equation in (21). The equations of motion for the surface variables are derived in a similar manner and are identical in appearance to the equations of motion (6) in the absence of vortices in the interior. However the interpretation of ϕ is now different, in that it must take into account the interior vortices. More precisely, in order to obtain Φ from ϕ, we must as before solve the Laplace equation for Φ with $\Phi = \phi$ on $y = \zeta$, however now we seek a different solution of the Laplace equation, namely the one with appropriate singularities at the positions of the vortices, so that the resulting velocity field $\mathbf{u} = \nabla\Phi$ has circulation Γ_α around the αth vortex. This, of course, influences the resulting surface velocity $\mathbf{u}^s$ and is the mechanism by which the vortices couple to the surface. In practice one would solve Laplace's equation for $\Lambda = \Phi - \Phi_p$, with surface value λ and $\Lambda \to 0$ as $y \to \infty$, where, as noted above, Λ has no singularities in the interior of the fluid.

For a single point vortex interacting with a free surface, a Froude number

$$F = \frac{\Gamma^2}{Y^3 g} \tag{22}$$

can be defined where Y is the initial depth. By scaling space and time, one can replace Γ, g, and Y with appropriate powers of F. Then $F \to 0$ (or $g \to \infty$) is a simple limit in which the free surface looks like a rigid lid and the potential Λ is the potential for the point together with its image.

B. Hamiltonian structure

In this section we show that the vortex-free surface system is Hamiltonian with respect to the following canonical brackets

$$\{\zeta(x), \phi(x')\} = \delta(x - x'), \quad \{X_\alpha, Y_\beta\} = \frac{\delta_{\alpha\beta}}{\Gamma_\alpha} \tag{23}$$

and a Hamiltonian which is the "excess energy" of the fluid. The excess energy must be used, since the presence of the vortices makes the kinetic energy of the fluid infinite. This infinity is due to two sources, the first being the $1/\epsilon$ divergence in the velocity as one approaches within a distance ϵ of a vortex, which results in an infinite self-energy for each vortex. The second source is the slow $1/R$ dropoff in the far field velocity as $R \to \infty$, in the case of nonzero net vorticity. Both of these divergences are logarithmic, and may be removed in a similar manner to the case in which one has point vortices only and no free surface. We present below formulas for the excess energy H, which is our Hamiltonian, and its variation with respect to the canonical variables $(\phi, \zeta, X_\alpha, Y_\alpha)$; later we discuss how these results are obtained. The Hamiltonian is given by

$$H = H_p + \tfrac{1}{2}\int_{-\infty}^{\infty} dx\{(\phi - \phi_p)[u_2^s + U_2^s - \zeta_x(u_1^s + U_1^s)] + \psi(U_1^s + \zeta_x U_2^s) + g\zeta^2\}, \tag{24}$$

where H_p is the usual point vortex Hamiltonian given in Eq. (2), and ψ is the stream function associated with the point-vortex velocity evaluated on the free surface,

$$\psi(x) = -\sum_\alpha \frac{\Gamma_\alpha}{2\pi} \ln|\mathbf{x} - \mathbf{X}^\alpha| \Big|_{y=\zeta(x,t)} . \tag{25}$$

The variation of this Hamiltonian with respect to the canonical variables is given by

$$\delta H = \sum_\alpha \Gamma_\alpha[v_1^\alpha(\mathbf{X}^\alpha)\delta Y_\alpha - v_2^\alpha(\mathbf{X}^\alpha)\delta X_\alpha] + \tfrac{1}{2}\int_{-\infty}^{\infty} dx \left[(u_2^s - u_1^s\zeta_x)\delta\phi + \left(\frac{|\mathbf{u}^s|^2}{2} + g\zeta - u_2^s(u_2^s - u_1^s\zeta_x)\right)\delta\zeta\right]. \tag{26}$$

It can therefore be seen immediately that Hamilton's equations resulting from the Poisson brackets (23) and Hamiltonian (24),

$$\zeta_t = \frac{\delta H}{\delta\phi}, \quad \phi_t = -\frac{\delta H}{\delta\zeta}, \tag{27}$$

$$\frac{dX_\alpha}{dt} = \frac{1}{\Gamma_\alpha}\frac{\partial H}{\partial Y_\alpha}, \quad \frac{dY_\alpha}{dt} = -\frac{1}{\Gamma_\alpha}\frac{\partial H}{\partial X_\alpha}, \tag{28}$$

are equivalent to the free surface plus vortex Eqs. (21), given in the preceding section.

Now we discuss how the Hamiltonian (24) and its variation (26) are computed. The Hamiltonian may be obtained by applying Green's theorem to Eq. (9), taking care that the boundary now includes branch cuts from the positions of the vortices to $y = -\infty$. An equivalent alternative approach which we describe now involves introducing a regularized energy functional $H_{\epsilon,R}$, parametrized by the pair (ϵ, R) given by

$$H_{\epsilon,R} = K_{\epsilon,R} + \int_{-\infty}^{\infty} dx \frac{g\zeta^2}{2}, \tag{29}$$

thus the potential energy is unchanged but the kinetic energy is now

$$K_{\epsilon,R} = \int d^2x \frac{|\mathbf{u}|^2}{2}\Theta(\zeta - y)\theta(R - |\mathbf{x}|)\prod_\alpha \Theta(|\mathbf{x} - \mathbf{X}^\alpha| - \epsilon), \tag{30}$$

where we have found it convenient to define the region of integration through the use of the Heaviside step function: $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$. Thus the square of the velocity field is integrated over a region bounded by the fluid surface above, excluding a circular region of radius ϵ around each vortex, and a distance less than R from the origin of the coordinate system. It is now not difficult to show by applications of Green's theorem, that for large R and small ϵ we have

$$H_{\epsilon,R} = C_1 \ln\epsilon + C_2 \ln \mathrm{R} + \mathrm{H} + O(\epsilon, R^{-1}), \tag{31}$$

where C_1 and C_2 are *constants* depending on the vortex circulations only, and H is the desired excess energy, which is independent of ϵ and R and is given in Eq. (24) above. Note that the Hamiltonian (24) is now finite for reasonable boundary conditions on ϕ and ζ as $x \to \pm\infty$.

Now because of the presence of $\mathbf{u}_s$ in H, our Hamiltonian is a highly implicit function of the canonical variables, and this is the reason we introduced $H_{\epsilon,R}$. The only easy way to compute the variation δH of the Hamiltonian with respect to the canonical variables (which is needed to arrive at the equations of motion) is to compute $\delta H_{\epsilon,R}$ to zeroth order in ϵ and R^{-1}. The advantage of $H_{\epsilon,R}$ compared to H is that it is defined by an integration over the fluid bulk, which facilitates the evaluation of the variations. To summarize, we are using the fact that from Eq. (31) we have $\delta H_{\epsilon,R} = \delta H + O(\epsilon, R^{-1})$, so that $\delta H_{\epsilon,R} \to \delta H$ as $\epsilon \to 0$ and $R \to \infty$. The actual computation of the variation $\delta H_{\epsilon,R}$ is rather technical and is presented in Appendix A. The result of the computation is, of course, Eq. (26), giving the variation of the excess energy H, in terms of the canonical variables.

Finally in this section we mention that besides the energy, the x component of the momentum is also conserved due to the translational symmetry of the Hamiltonian in the horizontal direction. It is a simple matter to show that this momentum (which is the generator of translations in x) is given by

$$P = \int d^2x\, u_1 \Theta(\zeta - y) = \sum_\alpha \Gamma_\alpha Y_\alpha + \int dx\, \zeta \phi_x . \tag{32}$$

III. FREE SURFACE MOTION IN THREE DIMENSIONS WITH VORTEX FILAMENTS IN THE INTERIOR

Next we discuss a generalization of the preceding ideas to three-dimensional motion. Formally, free surface motion with potential flow in the interior is handled in the same way as the two-dimensional case. That is, the fluid energy acts as a Hamiltonian, with the free surface elevation $z = \zeta(x, y, t)$ and the velocity potential evaluated on the free surface $\phi(x, y, t) = \Phi(x, y, \zeta, t)$ acting as canonical variables; where Φ is the velocity potential for the interior flow as before. However, if we wish to include vorticity in the interior, in a manner analogous to point vortices in two dimensions, the problem becomes quite a bit more complicated. Unfortunately there are no analogues of "point" vortices in three dimensions, in the sense of vortex singularities concentrated on points. A point vortex in two dimensions actually corresponds to a straight vortex filament in three dimensions, of course. Therefore our analogue for point vortices in three-dimensional motion will be vortex filaments which, however, must be in general, curved. This will introduce a mathematical difficulty and a physical difficulty. The mathematical difficulty is the fact that unlike the point-vortex problem in two dimensions, there do not appear to be canonical variables for filament motion. In the section below we solve this difficulty by showing that locally one can find canonical variables for this problem. The physical difficulty is due to the fact that an infinitely thin *curved* vortex filament induces a *normal* velocity on itself, which is infinite. Thus one must give the vortex filament a finite width to avoid this problem. As we shall indicate later, the Hamiltonian formalism is very well suited to dealing with such procedures in a systematic manner. Thus we will encounter many divergent quantities, and we shall proceed with the understanding that for practical work, certain regularizations must be effected, some of which are similar to those we performed in two dimensions. One regularization has to do with giving the filaments a finite core size, in order to obtain a nondivergent self-induced velocity. This regularization has been discussed extensively, albeit in a non-Hamiltonian context (see the review articles by Leonard [11], and Shariff and Leonard [15]). If the vortex filaments have finite length (which means that they must be closed) then the above regularization will also make the Hamiltonian finite. It is easy to include some of these regularization methods in our Hamiltonian framework. This issue will be addressed in a later section.

A. Vortex filaments

In this section we discuss canonical coordinates and Hamiltonian structure for a three-dimensional fluid with vorticity concentrated on one-dimensional filaments, in the absence of free surfaces. In the next section, we will combine the canonical coordinates for free surface plus vortex filament motion, in the spirit of the work above on two-dimensional free surface plus point-vortex motion, and demonstrate explicitly that the combined system is Hamiltonian.

There are two steps in demonstrating that a system can be put into canonical-Hamiltonian form. One must first find canonical Poisson brackets and then one must verify that the bracket structure gives the correct equations of motion. Finding a correct set of Poisson brackets can be done by intelligent guessing or by using the methods described by Marsden and Weinstein. The first example that we discuss is simple enough that the formalism of Marsden and Weinstein, which is rather technical, may be avoided. The details of a systematic derivation using their methods are presented in Appendix B.

For simplicity, we will work on the case where one has a single vortex filament. The generalization to several filaments is straightforward and we shall comment on it at the end of this section. It has already been shown by Marsden and Weinstein [14] that vortex filament motion is Hamiltonian in the sense that the motion occurs in a symplectic manifold, with the kinetic energy of the system acting as Hamiltonian. This symplectic manifold, which is infinite dimensional, is simply the space of non-self-intersecting curves of given fixed topology in the three-dimensional region occupied by the fluid. (Since, for example, a knotted vortex filament cannot evolve into one without knots in an inviscid fluid.) Marsden and Weinstein provide a formula for the symplectic two form on this manifold, and do no not discuss Poisson brackets and canonical coordinates. It must be stressed that this manifold of closed curves is genuinely curved and one can only expect to provide local canonical coordinates on it.

In order to set the stage, we shall discuss one case in which one can make a reasonable guess at the Poisson brackets. Suppose a vortex filament of circulation Γ is shaped so that one can describe it by the conditions

$$y = Y(x), \quad z = Z(x). \tag{33}$$

Then each plane x =const is pierced by the filament exactly once, and at each x we have a situation reminiscent of the two-dimensional problem. We will show that these are canonical coordinates for the problem

$$\{Y(x), Y(x')\} = \{Z(x), Z(x')\} = 0, \\ \{Y(x), Z(x')\} = \frac{\delta(x - x')}{\Gamma}. \tag{34}$$

We will actually treat a somewhat more general situation, which includes the above Cartesian case, as a special case. Of course the fact that later we will obtain the correct equations of motion from these brackets provides an independent confirmation of their correctness. However, for the sake of clarity we first look at this special case in some detail. In the absence of a free surface, if description of the filament in terms of coordinates given in Eq. (33) is appropriate, then the Hamiltonian is given by

$$H_F = \frac{\Gamma^2}{8\pi} \int dx\, dx' \frac{\mathbf{t}(x) \cdot \mathbf{t}(x')}{\{(x - x')^2 + [Y(x) - Y(x')]^2 + [Z(x) - Z(x')]^2\}^{1/2}}, \tag{35}$$

where $\mathbf{t}(x) = [1, Y'(x), Z'(x)]$ is the tangent vector (not normalized) to the filament. The motion of the filament is then governed by Hamilton's canonical equations,

$$\Gamma \frac{\partial Y(x,t)}{\partial t} = \frac{\delta H_F}{\delta Z(x,t)}, \\ \Gamma \frac{\partial Z(x,t)}{\partial t} = -\frac{\delta H_F}{\delta Y(x,t)}. \tag{36}$$

A more complete derivation of the free surface plus vortex filament motion given later will include the above as a special case. For now we note that usually the filament cannot be described by the parametrization given in Eq. (33). However, such a parametrization always works locally, and one can divide the curve into several sections with a different parametrization for each section.

As mentioned above we will actually treat a more general situation. Suppose that the curve of the vortex filament can be described in the following way. Let (a, b, c) be a fixed right-handed orthogonal coordinate system in three-dimensional space. We shall assume that our vortex filament is concentrated on a curve which is described by the condition

$$(a, b, c) = (a, B(a), C(a)). \tag{37}$$

These are indeed *local* coordinates, since not all vortex filaments will be describable in this fashion if (a, b, c) are a given and fixed. There is no guarantee either that a filament described in this manner initially can be described in this way at a later time. We will, for now, put these objections aside and consider $(B(a), C(a))$ as local coordinates for the vortex filament system. Although we have not yet looked in detail at the situation when description in terms of a given coordinate system breaks down, it appears that there is no difficulty in using different local coordinates for different segments of a filament and changing from one set to another as the system evolves in time. We will make a few remarks about this situation at the end of this section.

Our goal now will be to compute the Poisson brackets $\{B(a), B(a')\}$, $\{B(a), C(a')\}$, $\{C(a), C(a')\}$. As mentioned above we will give only the result of this computation and the details will be left to an appendix. Let τ be the Jacobian determinant of the transformation from the Cartesian (x, y, z) to the curvilinear (a, b, c), evaluated on the filament,

$$\tau = \tau(a) = \det \left. \frac{\partial(x, y, z)}{\partial(a, b, c)} \right|_{(a,b,c)=(u,B(a),C(a))}. \tag{38}$$

With this, the Poisson brackets of interest can be written as

$$\{B(a), B(a')\} = \{C(a), C(a')\} = 0, \\ \{B(a), C(a')\} = \frac{1}{\Gamma \tau(a)} \delta(a - a'). \tag{39}$$

Note that due to the τ^{-1} factor in $\{B(a), C(a')\}$, these coordinates are not necessarily canonical. However they can be modified into canonical coordinates in some special cases of interest which we now discuss. If the filament is nearly straight, then we may want to use Cartesian coordinates to describe it, e.g., as,

$$(x, y, z) = (x, Y(x), Z(x)), \tag{40}$$

here $\tau = 1$ and we *do* automatically have canonical coordinates

$$\{Y(x), Z(x')\} = \frac{\delta(x - x')}{\Gamma}, \tag{41}$$

which is a natural generalization of the point-vortex Poisson bracket. For a vortex ring a description in terms of cylindrical coordinates may be appropriate,

$$(\theta, z, r) = (\theta, Z(\theta), R(\theta)), \tag{42}$$

where the order of the coordinates is dictated by their right handedness. Here $\tau = R(\theta)$, therefore $Z(\theta)$ and $R^2(\theta)/2$ provide a canonical pair,

$$\{Z(\theta), R^2(\theta')/2\} = \frac{\delta(\theta - \theta')}{\Gamma}. \tag{43}$$

As a final example, for a filament of nearly helical shape, we may put

$$(r, \theta, z) = (r, \Theta(r), Z(r)), \tag{44}$$

then $\tau = r$ and we have the canonical pair $(r\Theta(r), Z(r))$,

$$\{r\Theta(r), Z(r')\} = \frac{\delta(r-r')}{\Gamma}. \tag{45}$$

The generalization of these Poisson brackets to the case of several filaments is straightforward. The Poisson bracket between variables of different filaments is zero. That is if we have N filaments described by

$$(a_\alpha, b_\alpha, c_\alpha) = (a_\alpha, B_\alpha(a_\alpha), C_\alpha(a_\alpha)), \quad \alpha = 1, \ldots, N \tag{46}$$

then,

$$\{B_\alpha(a_\alpha), B_\beta(a'_\beta)\} = \{C_\alpha(a_\alpha), C_\beta(a'_\beta)\} = 0,$$
$$\{B_\alpha(a_\alpha), C_\beta(a'_\beta)\} = \frac{\delta_{\alpha\beta}}{\Gamma\tau_\alpha(a_\alpha)}\delta(a_\alpha - a'_\alpha). \tag{47}$$

It should be noted that one can use different kinds of variables for different filaments, for instance, if we have a ring and a nearly straight filament, we may use cylindrical coordinates for the first and Cartesian coordinates for the second filament.

Finally, let us suppose that we give up the notion of describing a filament in terms of a single coordinate system. In this case, it appears reasonable to divide the filament into sections such that each section can be described in terms of a locally Cartesian pair $(Y(x), Z(x))$ or $(X(z), Y(z))$ or $(Y(y), X(y))$. Since the analysis giving our Poisson brackets is completely local, we will still have canonical brackets in each section of the filament. Therefore the only additional complication is that we have to keep track of how each section is connected with the adjacent pair. There must be a region around the point of separation such that either coordinate system could be used. As the system evolves in time, the description of a filament in terms of its segments may change. While this procedure may appear complicated, it may be no worse than more usual Lagrangian parametrizations of the filament, where the same kinds of problems will come up in another guise, if the filament becomes stretched and twisted enough.

B. Equations of motion for free surface motion with a vortex filament in the interior

Suppose now that we have a three-dimensional fluid with a free surface described by

$$z = \zeta(x, y, t), \tag{48}$$

and with a vortex filament in the interior described as in Eq. (37) of the preceding section. We have chosen to have only one filament in the interior, since the case of several filaments requires precisely the same arguments, but increases the notational clutter. In the end we will discuss the modifications which must be made to include more than one filament. We will now adopt a parallel notation to the first part of this paper on two-dimensional motion. Namely we will let $\mathbf{u}$, $\mathbf{U}$, and $\mathbf{v}$ stand for total velocity, velocity induced by the filament in an unbounded fluid, and the irrotational difference $\mathbf{u} - \mathbf{U}$, respectively. Let Φ, Φ_f, and Λ be the velocity potentials associated with these velocities, respectively, where the f subscript stands for filament. As before we denote the surface values of these potentials by lower case letters, e.g., $\phi(x,y,t) = \Phi(x,y,\zeta(x,y,t),t)$, while surface values of the velocities will be denoted by a superscript s, e.g., $\mathbf{u}^s(x,y,t) = \mathbf{u}(x,y,\zeta(x,y,t),t)$. The equations of motion for the system can then be written in the same way as before, i.e., the equations of motion for the surface variables are obtained by using kinematic and dynamic boundary conditions for the free surface, and the equations of motion for the vortex filaments are the statement that they move with the fluid. Thus the equations of motion for the surface variables can be written

$$\phi_t = u_3^s(u_3^s - u_1^s\zeta_x - u_2^s\zeta_y) - \frac{|\mathbf{u}^s|^2}{2} - g\zeta,$$
$$\zeta_t = u_3^s - u_1^s\zeta_x - u_2^s\zeta_y. \tag{49}$$

As in the two-dimensional case, the surface velocities are obtained by solving Laplace's equation for Φ with $\Phi = \phi$ at $z = \zeta$. The solution of the Laplace equation needed is the one with the appropriate singularities in the interior at the location of the filament. Some additional notation must be introduced before we write the filament equations. We define first the triad of orthogonal vector fields $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ associated with the orthogonal (a,b,c) coordinates in the usual way,

$$\mathbf{e}_a = \frac{\partial\mathbf{x}}{\partial a}, \quad \mathbf{e}_b = \frac{\partial\mathbf{x}}{\partial b}, \quad \mathbf{e}_c = \frac{\partial\mathbf{x}}{\partial c}, \tag{50}$$

with $(x,y,z) = \mathbf{x} = \mathbf{x}(a,b,c)$ being the Cartesian coordinates expressed in terms of (a,b,c). This orthogonal system is not normalized, however it is actually more convenient for our purposes to work with the non-normalized set. The position of the vortex filament as parametrized by a will be denoted $\mathbf{X}(a) = \mathbf{x}(a, B(a), C(a))$. The total velocity field $\mathbf{u}$ can be expanded in terms of the orthogonal vector fields $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$,

$$\mathbf{u} = u_a\mathbf{e}_a + u_b\mathbf{e}_b + u_c\mathbf{e}_c, \tag{51}$$

where $u_a = \mathbf{u}\cdot\mathbf{e}_a/|e_a|^2$ and so on. The equations of motion for the filament, i.e., equations for $B(a,t)$ and $C(a,t)$, can be obtained from

$$u_b = \frac{db}{dt} = \frac{\partial B}{\partial t} + \frac{da}{dt}\frac{\partial B}{\partial a}, \tag{52}$$

with a similar equation for u_c. This gives the following equations of motion for B and C:

$$\frac{\partial B}{\partial t} = u_b(\mathbf{X}(a)) - \frac{\partial B}{\partial a}u_a(\mathbf{X}(a)),$$
$$\frac{\partial C}{\partial t} = u_c(\mathbf{X}(a)) - \frac{\partial C}{\partial a}u_a(\mathbf{X}(a)). \tag{53}$$

Therefore in Eqs. (49) and (53), we have a set of coupled equations for the motion of the surface with a vortex filament in the interior. If one has N filaments in the interior described as in Eq. (46), then the only modification to the above equations is that the solution of the Laplace equation for Φ must take into account singularities due to all filaments in the interior; and the pair of equations (53) are replaced by the $2N$ equations which are obtained simply by giving an index α to each of a, B, C and $\mathbf{X}$, for

$\alpha = 1, \dots, N$. We reiterate that different coordinate systems can be used for each filament if needed.

C. Hamiltonian structure of the combined free surface and filament system

In this section we will show that the combined free surface and filament equations of motion (49) and (53) can be written as a Hamiltonian system with the Hamiltonian given by the sum of kinetic and potential energies

$$H = K + P = \tfrac{1}{2}\int d^3x\,|\mathbf{u}|^2\Theta(\zeta - z) + \frac{g}{2}\int dx\,dy\,\zeta^2. \tag{54}$$

The Hamiltonian expressed in terms of surface variables is given in Eq. (75) below. Poisson brackets are given by the combination of the free surface and vortex filament Poisson brackets,

$$\{\zeta(x,y),\phi(x',y')\} = \delta(x-x')\delta(y-y'),$$
$$\{B(a),C(a')\} = \frac{1}{\Gamma\tau(a)}\delta(a-a'). \tag{55}$$

Note that again the kinetic energy involves an integration over the fluid bulk while the potential energy is expressed as an integral over the free surface. Our task now is to compute the variation δH of the Hamiltonian with respect to these variables.

Now we turn to the task of computing the variation of the Hamiltonian with respect to the variables (ζ,ϕ,B,C). The potential energy is easy to deal with,

$$\delta P = g\int dx\,dy\,\zeta\delta\zeta. \tag{56}$$

We start the computation of the variation of the kinetic energy in the same way as the two-dimensional case,

$$\begin{aligned}\delta K &= \tfrac{1}{2}\int d^3x\,|\mathbf{u}|^2\delta\zeta\delta(\zeta - z) + \int d^3x\,\mathbf{u}\cdot\delta\mathbf{u}\,\Theta(\zeta - z)\\ &= \tfrac{1}{2}\int dx\,dy\,|\mathbf{u}^s|^2\delta\zeta \;+ \int d^3x\,\Theta(\zeta - z)\mathbf{u}\cdot(\nabla\delta\Lambda + \delta\mathbf{U})\\ &\equiv \delta K_1 + \delta K_2,\end{aligned} \tag{57}$$

where δK_1 and δK_2 are defined to be the first and second terms in the right-hand side of the line above. δK_1 is already in the desired form, so we proceed to compute δK_2. Now $\mathbf{U}$ is related to the vorticity $\boldsymbol{\omega}$ by the Biot-Savart law

$$\begin{aligned}\mathbf{U}(\mathbf{x}) &= \frac{1}{4\pi}\boldsymbol{\nabla}\times\int d^3x'\,\frac{\boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ &= \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\times\int\frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|},\end{aligned} \tag{58}$$

where the second expression, valid for a vortex filament of circulation Γ, is a path integral along the filament. In terms of our parametrization of the filament, this may also be written

$$\mathbf{U}(\mathbf{x}) = \boldsymbol{\nabla}\times\Psi(\mathbf{x}), \tag{59}$$

where $\Psi(\mathbf{x})$ is the vector potential given by

$$\Psi(\mathbf{x}) = \frac{\Gamma}{4\pi}\int da\frac{\mathbf{t}(a)}{|\mathbf{x}-\mathbf{X}(a)|}, \tag{60}$$

with

$$\mathbf{t}(a) = \frac{d\mathbf{X}(a)}{da} = \mathbf{e}_a + B'(a)\mathbf{e}_b + C'(a)\mathbf{e}_c \tag{61}$$

being the tangent vector along the filament. Note that $\mathbf{t}$ is *not* necessarily a unit tangent. The dependence of $\mathbf{U}$ on our variables (ζ,ϕ,B,C) is through $\mathbf{X}(a)$ (which depends on B and C only). So starting from Eq. (59), we have

$$\delta\mathbf{U}(\mathbf{x}) = \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\times\int da\left[\frac{1}{|\mathbf{x}-\mathbf{X}(a)|}\frac{d}{da}\delta\mathbf{X}(a) + \mathbf{t}(a)\delta\left(\frac{1}{|\mathbf{x}-\mathbf{X}(a)|}\right)\right]. \tag{62}$$

Integrating by parts in the first term and using

$$\delta\left(\frac{1}{|\mathbf{x}-\mathbf{X}(a)|}\right) = \frac{[\mathbf{x}-\mathbf{X}(a)]\cdot\delta\mathbf{X}(a)}{|\mathbf{x}-\mathbf{X}(a)|^3}, \tag{63}$$

in the second term, we arrive at

$$\begin{aligned}\delta\mathbf{U}(\mathbf{x}) &= \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\times\int da\frac{\mathbf{t}(a)[\mathbf{x}-\mathbf{X}(a)]\cdot\delta\mathbf{X}(a) - \delta\mathbf{X}(a)[\mathbf{x}-\mathbf{X}(a)]\cdot\mathbf{t}(a)}{|\mathbf{x}-\mathbf{X}(a)|^3}\\ &= \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\times\int da\frac{[\delta\mathbf{X}(a)\times\mathbf{t}(a)]\times[\mathbf{x}-\mathbf{X}(a)]}{|\mathbf{x}-\mathbf{X}(a)|^3}\\ &= \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int da\frac{\delta\mathbf{X}(a)\times\mathbf{t}(a)}{|\mathbf{x}-\mathbf{X}(a)|^3},\end{aligned} \tag{64}$$

now using $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$, and the fact that $\nabla^2|\mathbf{x}-\mathbf{x}'|^{-1} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$, we get

$$\delta\mathbf{U}(\mathbf{x}) = \Gamma\int da\,\delta^3(\mathbf{x}-\mathbf{X}(a))\delta\mathbf{X}(a)\times\mathbf{t}(a) + \frac{\Gamma}{4\pi}\boldsymbol{\nabla}\int da\,\boldsymbol{\nabla}\cdot\left(\frac{\delta\mathbf{X}(a)\times\mathbf{t}(a)}{|\mathbf{x}-\mathbf{X}(a)|}\right). \tag{65}$$

Inserting this expression into δK_2 in Eq. (57) above, we have

$$\delta K_2 = \Gamma \int da\, \mathbf{u}(\mathbf{X}(a)) \cdot (\delta\mathbf{X}(a) \times \mathbf{t}(a)) + \int d^3x\, \Theta(\zeta - z)\mathbf{u} \cdot \boldsymbol{\nabla} \left[\delta\Lambda + \frac{\Gamma}{4\pi} \int da\, \boldsymbol{\nabla} \cdot \left(\frac{\delta\mathbf{X}(a) \times \mathbf{t}(a)}{|\mathbf{x} - \mathbf{X}(a)|}\right)\right]$$
$$\equiv \delta K_{21} + \delta K_{22}, \tag{66}$$

where δK_{21} and δK_{22} stand for the two terms in the line above, respectively.

We discuss first δK_{21}. We expand each of $\mathbf{u}(\mathbf{X}(a))$, $\mathbf{t}(a)$, and $\delta\mathbf{X}(a)$ in terms of the orthogonal set $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$, given in Eq. (50). In the case of $\mathbf{u}(\mathbf{X}(a))$ we just have to evaluate the expression (51) along the filament

$$\mathbf{u}(\mathbf{X}(a)) = u_a(\mathbf{X}(a))\mathbf{e}_a + u_b(\mathbf{X}(a))\mathbf{e}_b + u_c(\mathbf{X}(a))\mathbf{e}_c. \tag{67}$$

The appropriate expression for the tangent vector $\mathbf{t}$ was already given in Eq. (61), while for $\delta\mathbf{X}(a)$ we have

$$\delta\mathbf{X}(a) = \delta B \frac{\partial \mathbf{x}}{\partial b} + \delta C \frac{\partial \mathbf{x}}{\partial c} = \delta B \mathbf{e}_b + \delta C \mathbf{e}_c. \tag{68}$$

Now using the fact that along the filament we have $\mathbf{e}_a \cdot (\mathbf{e}_b \times \mathbf{e}_c) = \tau$, with τ being the Jacobian determinant along the filament given in Eq. (38), we can express δK_{21} in terms of variations of our coordinates (ζ, ϕ, B, C)

$$\delta K_{21} = \Gamma \int da\, \tau(a) \left[\left(u_b(\mathbf{X}(a)) - u_a(\mathbf{X}(a))\frac{\partial B}{\partial a}\right)\delta C(a) - \left(u_c(\mathbf{X}(a)) - u_a(\mathbf{X}(a))\frac{\partial C}{\partial a}\right)\delta B(a)\right]. \tag{69}$$

It now remains to compute δK_{22} in Eq. (66) in terms of variations of (ζ, ϕ, B, C). To this end we first integrate by parts, and use $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \boldsymbol{\nabla}\Theta(\zeta - z) = \delta(\zeta - z)(u_1\zeta_x + u_2\zeta_y - u_3)$, in order to arrive at the following expression for δK_{22}

$$\delta K_{22} = \int dx\, dy\, (u_3^s - u_1^s\zeta_x - u_2^s\zeta_y) \left\{(\delta\Lambda)_s + \frac{\Gamma}{4\pi}\left[\int da\, \boldsymbol{\nabla} \cdot \left(\frac{\delta\mathbf{X}(a) \times \mathbf{t}(a)}{|\mathbf{x} - \mathbf{X}(a)|}\right)\right]_s\right\}, \tag{70}$$

where as before the subscript or superscript s denotes evaluation of the expression on the free surface, $(x, y, z) = (x, y, \zeta)$. Now, in a manner analogous to the two-dimensional case, the complicated expression occurring after $(\delta\Lambda)_s$ turns out simply to be the variation of the velocity potential of the filament, i.e.,

$$(\delta\Phi_f)_s = \frac{\Gamma}{4\pi}\left[\int da\, \boldsymbol{\nabla} \cdot \left(\frac{\delta\mathbf{X}(a) \times \mathbf{t}(a)}{|\mathbf{x} - \mathbf{X}(a)|}\right)\right]_s. \tag{71}$$

We have chosen to present the involved proof of Eq. (71) in Appendix C, so as not to distract the reader from the main thrust of the argument, which is now quite similar to the two-dimensional case. In any case, with Eq. (71), we can now write

$$\delta K_{22} = \int dx\, dy\, (u_3^s - u_1^s\zeta_x - u_2^s\zeta_y)(\delta\phi - u_3^s\delta\zeta), \tag{72}$$

where with the same reasoning as in the two-dimensional case we have used

$$(\delta\Lambda)_s + (\delta\Phi_f)_s = (\delta\Phi)_s = \delta\phi + u_3^s\delta\zeta. \tag{73}$$

Now putting together Eqs. (54), (56), (57), (66), (69), and (72), we arrive at the following expression for the variation of the Hamiltonian (54) in terms of variations of the coordinates (ζ, ϕ, B, C):

$$\delta H = \Gamma \int da\, \tau(a) \left[\left(u_b(\mathbf{X}(a)) - u_a(\mathbf{X}(a))\frac{\partial B}{\partial a}\right)\delta C(a) - \left(u_c(\mathbf{X}(a)) - u_a(\mathbf{X}(a))\frac{\partial C}{\partial a}\right)\delta B(a)\right]$$
$$+ \int dx\, dy \left[(u_3^s - u_1^s\zeta_x - u_2^s\zeta_y)\delta\phi - \left(u_3^s(u_3^s - u_1^s\zeta_x - u_2^s\zeta_y) - \frac{|\mathbf{u}_s|^2}{2} - g\zeta\right)\delta\zeta\right]. \tag{74}$$

With this expression for δH, it can readily be observed that as claimed, the Hamiltonian (54) and Poisson brackets (55) lead to the free surface and vortex filament equations of motion given in Eqs. (49) and (53).

D. Energy, momentum, and angular momentum

In the preceding section we computed the variation of the vortex filament plus free surface Hamiltonian δH in terms of our variables (B, C, ζ, ϕ). It is desirable to have an expression for H itself [given in Eq. (54)], in terms of these variables. This can be accomplished by appropriate integration by parts in (54). The result is given by

$$H = \frac{\Gamma^2}{8\pi} \int da\, da' \frac{\mathbf{t}(a) \cdot \mathbf{t}(a')}{|\mathbf{X}(a) - \mathbf{X}(a')|}$$
$$+ \tfrac{1}{2} \int dx\, dy\, [\boldsymbol{\psi} \cdot (\mathbf{U}^s \times \mathbf{n}) + (\phi - \phi_f)(\mathbf{u}^s + \mathbf{U}^s) \cdot \mathbf{n}], \tag{75}$$

where $\boldsymbol{\psi}$ is the vector potential for the vortex filament velocity, evaluated on the surface, i.e., $\boldsymbol{\psi}(x,y) = \Psi(x,y,\zeta)$. Similarly ϕ_f is the filament velocity potential evaluated on the free surface. An expression for the filament velocity potential is given in Eq. (C1) of Appendix C. Finally $\mathbf{n}$ is the normal vector to the surface, $\mathbf{n} = (-\zeta_x, -\zeta_y, 1)$. Note that our Hamiltonian consists of two parts. The first term, which is an integral over the filament, is the appropriate Hamiltonian for the motion of a filament without a free surface. The second term is an integral over the free surface, which reduces to the usual irrotational free surface Hamiltonian when the filament is absent.

In this problem in addition to the energy, the horizontal components of the momentum P_1 and P_2 are conserved due to the translational symmetry of the system in the x and y directions. Additionally, the z component of the angular momentum L_3 is conserved, reflecting rotational symmetry about the z axis. These quantities are given by

$$
\begin{aligned}
P_1 &= \int d^3x\, \Theta(\zeta - z) u_1, \\
P_2 &= \int d^3x\, \Theta(\zeta - z) u_2, \\
L_3 &= \int d^3x\, \Theta(\zeta - z)(x u_2 - y u_1).
\end{aligned} \tag{76}
$$

As in the case of the Hamiltonian, it is desirable to express these in terms of our chosen coordinates (B, C, ζ, ϕ). This is again accomplished by integration by parts with the use of suitable integrating factors. We present the results below

$$
\begin{aligned}
P_1 &= \int dx\, dy\, \zeta \phi_x - \Gamma \int da\, Z(a) Y'(a), \\
P_2 &= \int dx\, dy\, \zeta \phi_y + \Gamma \int da\, Z(a) X'(a), \\
L_3 &= \tfrac{1}{2} \int dx\, dy\, (x^2 + y^2)(\phi_x \zeta_y - \phi_y \zeta_x) \\
&\quad - \frac{\Gamma}{2} \int da\, Z'(a)[X^2(a) + Y^2(a)].
\end{aligned} \tag{77}
$$

E. Regularization

It is well known that vortex filaments have an infinite self-induced velocity. A variety of ways have been suggested for regularizing the motion (see, for example, Leonard [11], and Sharif and Leonard [15]). Some, but not all, regularization methods can be included in a Hamiltonian framework. We describe two simple examples in order to give an idea of how one proceeds.

First, one can simply modify the denominator in the first term of the Hamiltonian (75),

$$
|\mathbf{X}(a) - \mathbf{X}(a')| \longrightarrow |\mathbf{X}(a) - \mathbf{X}(a') + \sigma^2|. \tag{78}
$$

The cutoff σ represents a "core radius." The equations of motion are derived as before, but now the self-induced velocity is finite. Formally our new regularized Hamiltonian H^R is given by

$$
H^R = (H - H_F) + H_F^R, \tag{79}
$$

where H is the old Hamiltonian, H_F is the filament part of the old Hamiltonian, and H_F^R is the filament Hamiltonian modified by the inclusion of core radius σ. Written as above we see that clearly H^R leads to a finite self-induced velocity and also that its variation δH_R needed to obtain the regularized equations of motion can be computed in a similar manner to the one used above for δH. It must be noted that since our description of the filament is not the same as the usual one [11, 15], the regularized equations are not identical to the usual equations obtained by directly modifying the self-induced velocity by addition of σ^2 in the denominator of the Biot-Savart law.

In a different spirit, we have the rather drastic method of regularization given by the local induction approximation (see Leonard [11], and references therein). This is very easy to describe in the Hamiltonian framework. We start with the filament Hamiltonian,

$$
H_F = \frac{\Gamma^2}{8\pi} \int dx\, dx' \, \frac{\mathbf{t}(x) \cdot \mathbf{t}(x')}{|\mathbf{X}(x) - \mathbf{X}(x')|}, \tag{80}
$$

and for each fixed x we suppose that the only contribution in the x' integral comes from x' taken in the range $x - \Delta < x' < x + \Delta$, where Δ is small. Then it is easy to show that to lowest order in Δ, we have

$$
H_F = \frac{\ln(\Delta)\Gamma^2}{4\pi} \int dx\, |\mathbf{t}(x)|. \tag{81}
$$

The resulting canonical equations of motion are easily obtained,

$$
\Gamma \frac{\partial Y}{\partial t} = \frac{\delta H_F}{\delta Z} = -\frac{\ln(\Delta)\Gamma^2}{4\pi} \frac{\partial}{\partial x} \left(\frac{Z(x)}{|\mathbf{t}(x)|} \right), \tag{82}
$$

with a similar equation for Z. It can readily be verified that these are equivalent to the usual local induction approximation equations. Of course, if the parametrization of the filament in terms of $Y(x)$ and $Z(x)$ fails, then separate equations will be needed for other segments of the filament in terms of $X(y)$ and $Z(y)$ or $X(z)$ and $Y(z)$, with equations of motion identical in form to the equation for $Y(x)$ above.

IV. VARIATIONAL PRINCIPLE

One of the advantages of a Hamiltonian formulation is the fact that the equations of motion are derivable from a variational principle. For a discussion of this point, see the paper by Miles [5]. Approximate variational solutions have the property that the first order error vanishes, which often means that the solution is a better representation of the exact solution than would be obtained with similar approximations on the equations of motion. It also means the conservation laws for momentum and angular momentum that arise because of invariances of the Hamiltonian are still exactly conserved. The canonical form of Hamilton's principle for the two-dimensional fluid with point vortices is given by requiring that the following form be stationary with respect to variations of the canonical coordinates $(X_\alpha, Y_\alpha, \zeta, \phi)$:

$$S = \int dt \left[\Gamma \sum_\alpha \dot{X}_\alpha Y_\alpha + \int dx\, \phi \frac{\partial \zeta}{\partial t} - H \right], \tag{83}$$

where the Hamiltonian H is given in Eq. (24). An alternate form, which is noncanonical, with independent variables $(X_\alpha, Y_\alpha, \zeta, \lambda)$, is given by

$$S = \int dt \left[\sum_\alpha \dot{\mathbf{X}}_\alpha \cdot \int d^2x\, \mathbf{U}^\alpha \Theta(\zeta - y) + \int dx\, \phi \frac{\partial \lambda}{\partial t} - H \right], \tag{84}$$

where $\mathbf{U}^\alpha$ is the point-vortex velocity given by Eq. (11). It is easily verified that requiring that S be stationary with respect to independent variations of $(X_\alpha, Y_\alpha, \zeta, \lambda)$, yields the correct equations of motion. In this form the coupling between the surface and the point vortices is clearly exhibited. It may well be that for practical approximate calculation schemes it is easier to work with this latter form of the variational principle. We note that, canonical and noncanonical variational principles can be constructed for the three-dimensional free surface-vortex filament problem, in a similar manner.

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APPENDIX A

In this appendix we present a derivation of Eq. (26) for the variation of the two-dimensional point vortex plus free surface Hamiltonian. As mentioned in the text, this computation is carried out by calculating the variation of the regularized Hamiltonian given in Eq. (30), to zeroth order in ϵ. We have

$$\delta H_{\epsilon,R} = \delta K_{\epsilon,R} + g \int_{-\infty}^{\infty} dx\, \zeta \delta\zeta, \tag{A1}$$

thus the main difficulty is in the computation of $\delta K_{\epsilon,R}$ given by Eq. (30). In this definition of $K_{\epsilon,R}$, Eq. (30), we will leave out the $\Theta(R - |\mathbf{x}|)$ factor, which has no dynamical content and is either passively carried through all the calculations, or gives rise to terms which obviously vanish as $R \to \infty$. Thus with the understanding that the symbol δ refers to variations with respect to the canonical coordinates $(\mathbf{X}^\alpha, \zeta, \phi)$, we compute

$$\begin{aligned} \delta K_{\epsilon,R} = \int d^2x \Bigg[& \mathbf{u} \cdot \delta \mathbf{u}\, \Theta(\zeta - y) \prod_\alpha \Theta(|\mathbf{x} - \mathbf{X}^\alpha| - \epsilon) \\ & + \frac{|\mathbf{u}|^2}{2} \prod_\alpha \Theta(|\mathbf{x} - \mathbf{X}^\alpha| - \epsilon) \delta\Theta(\zeta - y) \\ & + \frac{|\mathbf{u}|^2}{2} \Theta(\zeta - y) \delta \prod_\alpha \Theta(|\mathbf{x} - \mathbf{X}^\alpha| - \epsilon) \Bigg] \\ \equiv {} & \delta K_1 + \delta K_2 + \delta K_3. \end{aligned} \tag{A2}$$

We will now deal separately with each of δK_1, δK_2, and δK_3. In order to compute δK_1 we note that

$$\mathbf{u} \cdot \delta \mathbf{u} = \mathbf{u} \cdot (\nabla \delta\Lambda + \delta \mathbf{U}), \tag{A3}$$

and

$$\delta \mathbf{U} = \delta \sum_\alpha \mathbf{U}^\alpha = \sum_\alpha \left(\frac{\partial \mathbf{U}^\alpha}{\partial X_\alpha} \delta X_\alpha + \frac{\partial \mathbf{U}^\alpha}{\partial Y_\alpha} \delta Y_\alpha \right). \tag{A4}$$

Next, one may use the fact that $\mathbf{U}^\alpha$ is a function of $\mathbf{x} - \mathbf{X}^\alpha$ to replace $\mathbf{X}^\alpha$ derivatives with $\mathbf{x}$ derivatives at the expense of a minus sign. Having done this, incompressibility of $\mathbf{U}^\alpha$ and the definition of vorticity as the curl of the velocity field may be used to obtain

$$\begin{aligned} \mathbf{u} \cdot \delta \mathbf{U} = \sum_\alpha \{ & \Gamma_\alpha \delta^2(\mathbf{x} - \mathbf{X}^\alpha)[u_1(\mathbf{x}) \delta Y_\alpha - u_2(\mathbf{x}) \delta X_\alpha] \\ & - \mathbf{u}(\mathbf{x}) \cdot \nabla (\mathbf{U}^\alpha \cdot \delta \mathbf{X}^\alpha) \}. \end{aligned} \tag{A5}$$

In the above equation and others to follow the symbol δ is used both for variations and the Dirac delta function [as in $\delta^2(\mathbf{x} - \mathbf{X}^\alpha)$], however the meaning should be clear from the context and should cause no confusion. Now since the region of integration in the defintion of $K_{\epsilon,R}$ explicity excludes the vortices, the first term in Eq. (A5) will not contribute to the variation, and we may thus conclude that, effectively

$$\mathbf{u} \cdot \delta \mathbf{u} = \mathbf{u} \cdot \nabla \left(\delta\Lambda - \sum_\alpha (\mathbf{U}^\alpha \cdot \delta \mathbf{X}^\alpha) \right). \tag{A6}$$

This relation may now be inserted into δK_1, which after an integration by parts and using $\nabla \cdot \mathbf{u} = 0$ becomes

$$\delta K_1 = \int d^2x \left(\sum_\alpha (\mathbf{U}^\alpha \cdot \delta \mathbf{X}^\alpha) - \delta\Lambda \right) \mathbf{u} \cdot \nabla \left(\theta(\zeta - y) \prod_\beta \Theta(|\mathbf{x} - \mathbf{X}^\beta| - \epsilon) \right). \tag{A7}$$

In the integrand above, the $\mathbf{u} \cdot \nabla$ operator acting on the Θ functions gives rise, by the product rule, to sums of products of δ functions and Θ functions. However where the resulting δ functions contribute (curves on which their arguments vanish), the Θ functions are equal to unity (i.e., their arguments are positive) therefore we obtain the following rather simple expression for the result of the differentiation:

$$\mathbf{u}\cdot\boldsymbol{\nabla}\left(\Theta(\zeta-y)\prod_\beta\Theta(|\mathbf{x}-\mathbf{X}^\beta|-\epsilon)\right)=(u_1\zeta_x-u_2)\delta(\zeta-y)+\sum_\beta\frac{\mathbf{u}\cdot(\mathbf{x}-\mathbf{X}^\beta)}{|\mathbf{x}-\mathbf{X}^\beta|}\delta(|\mathbf{x}-\mathbf{X}^\beta|-\epsilon). \tag{A8}$$

Therefore now δK_1 itself splits naturally into two terms, $\delta K_1=\delta K_1^1+\delta K_1^2$, with which we deal separately. First in δK_1^1, the δ function $\delta(\zeta-y)$ enables us to perform the integral over y and results in

$$\delta K_1^1=\int_{-\infty}^{\infty}dx\left((\delta\Lambda)_s-\sum_\alpha(\mathbf{U}_s^\alpha\cdot\delta\mathbf{X}^\alpha)\right)(u_2^s-u_1^s\zeta_x), \tag{A9}$$

where $(\delta\Lambda)_s$ denotes the variation of Λ evaluated on the free surface. Now we have

$$\begin{aligned}\sum_\alpha(\mathbf{U}_s^\alpha\cdot\delta\mathbf{X}^\alpha)&=\sum_\alpha\left(\frac{\partial\Phi_p^\alpha}{\partial\mathbf{x}}\right)_s\cdot\delta\mathbf{X}^\alpha\\&=-\sum_\alpha\left(\frac{\partial\Phi_p}{\partial\mathbf{X}^\alpha}\right)_s\cdot\delta\mathbf{X}^\alpha\\&=(\delta\Phi_p)_s.\end{aligned} \tag{A10}$$

Thus

$$(\delta\Lambda)_s-\sum_\alpha(\mathbf{U}_s^\alpha\cdot\delta\mathbf{X}^\alpha)=(\delta\Phi)_s. \tag{A11}$$

Now we must relate the variation of Φ evaluated on the surface, i.e., $(\delta\Phi)_s$, to variations of the surface variables ϕ and ζ. We have

$$\delta\phi=\delta(\Phi(x,\zeta))=[\delta\Phi(x,y)]_s+\left(\frac{\partial\Phi}{\partial y}\right)_s\delta\zeta, \tag{A12}$$

thus

$$(\delta\Phi)_s=\delta\phi-u_2^s\delta\zeta. \tag{A13}$$

With this we can now express δK_1^1 in terms of variations of the canonical variables, we record the result below,

$$\delta K_1^1=\int_{-\infty}^{\infty}dx\,(\delta\phi-u_2^s\delta\zeta)(u_2^s-u_1^s\zeta_x). \tag{A14}$$

We now turn our attention to δK_1^2. We have

$$\delta K_1^2=\sum_\beta\int d^2x\left(\sum_\alpha(\mathbf{U}^\alpha\cdot\delta\mathbf{X}^\alpha)-\delta\Lambda\right)\times\frac{\mathbf{u}\cdot(\mathbf{x}-\mathbf{X}^\beta)}{|\mathbf{x}-\mathbf{X}^\beta|}\delta(|\mathbf{x}-\mathbf{X}^\beta|-\epsilon). \tag{A15}$$

It is evident that due to the δ function in the integrand above, that as $\epsilon\to0$ the integration is over circular arcs whose perimeters are tending to zero. Therefore to zeroth order in ϵ (which is the contribution we are interested in), contributions are made only from terms which become singular as $\epsilon\to0$. In order to analyze this situation, we introduce polar coordinates (r,θ) centered around the vortices by writing

$$x=X_\beta+r\cos\theta,\quad y=Y_\beta+r\sin\theta. \tag{A16}$$

Transforming into these polar coordinates in δK_1^2, we may then perform the integral over r by using the δ function $\delta(|\mathbf{x}-\mathbf{X}^\beta|-\epsilon)=\delta(r-\epsilon)$, which gives

$$\delta K_1^2=\epsilon\sum_\beta\int_0^{2\pi}d\theta\left(\sum_\alpha(\mathbf{U}^\alpha\cdot\delta\mathbf{X}^\alpha)-\delta\Lambda\right)\times\mathbf{u}\cdot(\cos\theta,\sin\theta). \tag{A17}$$

In the expression above, note should be made of the following facts. First, that we now have an explicit ϵ factor, which came from $dx\,dy=r\,dr\,d\theta$ and the δ function and secondly, that the quantities $\mathbf{U}^\alpha$, $\delta\Lambda$, and $\mathbf{u}$ are now evaluated at $(X_\beta+\epsilon\cos\theta,Y_\beta+\epsilon\sin\theta)$, again because of the change in coordinates and the integration on r. Now in the expression for δK_1^2 above, we make the decomposition of the total velocity defined in Eq. (13), that is $\mathbf{u}=\mathbf{U}^\beta+\mathbf{v}^\beta$. But in these coordinates we have

$$\mathbf{U}^\beta(X_\beta+\epsilon\cos\theta,Y_\beta+\epsilon\sin\theta)=\frac{\Gamma_\beta}{2\pi\epsilon}(-\sin\theta,\cos\theta), \tag{A18}$$

which has zero dot product with $(\cos\theta,\sin\theta)$ in δK_1^2 above. Further, we make the following decomposition:

$$\sum_\alpha(\mathbf{U}^\alpha\cdot\delta\mathbf{X}^\alpha)=\mathbf{U}^\beta\cdot\delta\mathbf{X}^\beta+\sum_{\alpha\,(\neq\beta)}(\mathbf{U}^\alpha\cdot\delta\mathbf{X}^\alpha). \tag{A19}$$

Now we note that $\delta\Lambda$, the sum on $(\mathbf{U}^\alpha\cdot\delta\mathbf{X}^\alpha)$ with β excluded from the sum, and $\mathbf{v}^\beta$, are by construction nonsingular as $\epsilon\to0$. Therefore we may write

$$\delta K_1^2=\epsilon\sum_\beta\int_0^{2\pi}d\theta\;\mathbf{U}^\beta\cdot\delta\mathbf{X}^\beta\,\mathbf{v}^\beta\cdot(\cos\theta,\sin\theta)+O(\epsilon). \tag{A20}$$

Finally using the explicit expression (A18) for $\mathbf{U}^\beta$, and the fact that correct to order ϵ we have $\mathbf{v}^\beta(X_\beta+\epsilon\cos\theta,Y_\beta+\epsilon\sin\theta)=\mathbf{v}^\beta(X_\beta,Y_\beta)$, we may carry out explicitly the integral over θ in δK_1^2, in order to arrive at the following expression for the variation in terms of the canonical variables:

$$\delta K_1^2=\tfrac12\sum_\beta\Gamma_\beta[v_1^\beta(\mathbf{X}^\beta)\delta Y_\beta-v_2^\beta(\mathbf{X}^\beta)\delta X_\beta]+O(\epsilon). \tag{A21}$$

This concludes our calculation of $\delta K_1=\delta K_1^1+\delta K_1^2$. Our next step in computing $\delta K_{\epsilon,R}$ is to compute δK_2 [see Eq. (A2)]. This variation is quite simple, we use $\delta\theta(\zeta-y)=\delta(\zeta-y)\,\delta\zeta$, to arrive at

$$\delta K_2=\int_{-\infty}^{\infty}dx\,\frac{|\mathbf{u}^s|^2}{2}\,\delta\zeta. \tag{A22}$$

This leaves δK_3 to complete the computation of $\delta K_{\epsilon,R}$. This computation is quite similar to the one carried out for δK_1^2 above in that one uses polar coordinates centered around the vortices to pick up the zeroth order term in ϵ for the variation. Indeed, using

$$\delta\prod_{\alpha}\Theta(|\mathbf{x}-\mathbf{X}^{\alpha}|-\epsilon)$$

$$=-\sum_{\beta}\frac{(\mathbf{x}-\mathbf{X}^{\beta})\cdot\delta\mathbf{X}^{\beta}}{|\mathbf{x}-\mathbf{X}^{\beta}|}\delta(|\mathbf{x}-\mathbf{X}^{\beta}|-\epsilon), \quad (A23)$$

and introducing polar coordinates as before, we arrive at

$$\delta K_3=-\epsilon\sum_{\beta}\int_0^{2\pi}d\theta\,\frac{|\mathbf{u}|^2}{2}(\cos\theta,\sin\theta)\cdot\delta\mathbf{X}^{\beta}, \quad (A24)$$

where as before, $\mathbf{u}$ above is evaluated at $(X_\beta+\epsilon\cos\theta, Y_\beta+\epsilon\sin\theta)$. Up to now the expression for δK_3 is exact. However, we need only the zeroth order behavior in ϵ, so as before make the decomposition in Eq. (13), giving

$$|\mathbf{u}|^2=|\mathbf{U}^{\beta}|^2+2\mathbf{U}^{\beta}\cdot\mathbf{v}^{\beta}+|\mathbf{v}^{\beta}|^2. \quad (A25)$$

The term in $|\mathbf{v}^{\beta}|^2$ is nonsingular as $\epsilon\to 0$, so it is of order ϵ and we ignore it. Now making use of the expression (A18) for $\mathbf{U}^{\beta}$, we see that the term in $|\mathbf{U}^{\beta}|^2$ is actually of order $\epsilon\epsilon^{-2}=\epsilon^{-1}$. However it integrates to zero, as it must. This leaves the cross term, which gives the only zeroth order contribution. Again we replace $\mathbf{v}^{\beta}(X_\beta+\epsilon\cos\theta, Y_\beta+\epsilon\sin\theta)$ with $\mathbf{v}^{\beta}(\mathbf{X}^{\beta})$, which is correct to order ϵ. The resulting trigonometric integrals are then performed to yield

$$\delta K_3=\tfrac{1}{2}\sum_{\beta}\Gamma_\beta[v_1^{\beta}(\mathbf{X}^{\beta})\delta Y_\beta-v_2^{\beta}(\mathbf{X}^{\beta})\delta X_\beta]+O(\epsilon). \quad (A26)$$

We can now put together Eqs. (A1), (A2), (A14), (A21), (A22), (A26), in order to write an expression for $\delta H_{\epsilon,R}$ valid to zeroth order in ϵ. However, as argued above, this expression will be *exact* for the variation of H, the actual Hamiltonian for this problem, given in Eq. (24). So the string of equations above yield

$$\delta H=\sum_{\alpha}\Gamma_\alpha[v_1^{\alpha}(\mathbf{X}^{\alpha})\delta Y_\alpha-v_2^{\alpha}(\mathbf{X}^{\alpha})\delta X_\alpha]+\tfrac{1}{2}\int_{-\infty}^{\infty}dx\left[(u_2^s-u_1^s\zeta_x)\delta\phi+\left(\frac{|\mathbf{u}^s|^2}{2}+g\zeta-u_2^s(u_2^s-u_1^s\zeta_x)\right)\delta\zeta\right]. \quad (A27)$$

APPENDIX B: COMPUTATION OF THE VORTEX FILAMENT POISSON BRACKETS

In this appendix we provide a derivation of the Poisson brackets for vortex filaments in terms of curvilinear coordinates (a,b,c) as given in Eq. (39) of the text. The derivation is given using the formula provided by Marsden and Weinstein [14] for the relevant symplectic two form as described in the text. A symplectic two form is a bilinear, antisymmetric, nondegenerate, and closed (for a discussion of the meaning of these terms, see, for instance, Schutz [16]) functional of a pair of vectors tangent to the symplectic manifold at a given point. In the case of vortex filaments, the meaning of these tangent vectors is intuitively clear. Tangent vectors always signify infinitesimal displacements relating nearby points on a manifold. One moves from one vortex filament to a nearby one along a vector field defined along the filament, and normal to the filament. One uses normal vector fields, since a given displacement defines a unique normal vector field and vice versa. Let $\mathbf{n}$ and $\mathbf{m}$ be two such normal vector fields, then Marsden and Weinstein provide the following formula for Ω, the appropriate symplectic two form for three-dimensional vortex dynamics defined by

$$\Omega(\mathbf{n},\mathbf{m})=\int d^3x\,\boldsymbol{\omega}\cdot(\mathbf{n}\times\mathbf{m}), \quad (B1)$$

where ω is the vorticity. Of course since the vorticity is concentrated along the filament, one needs to provide the vector fields $\mathbf{n}$ and $\mathbf{m}$ only along the filament, which is their domain of definition. Before we proceed to Poisson brackets, we will realize the formula above for the symplectic two form more concretely in the (a,b,c) coordinates. Let $(\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c)$ be the triad of orthogonal vector fields associated with the orthogonal (a,b,c) as defined in Eq. (50). The vorticity of the filament can then be written,

$$\boldsymbol{\omega}(a,b,c)=\frac{\Gamma\mathbf{t}(a)}{J(a,b,c)}\delta(b-B(a))\delta(c-C(a)), \quad (B2)$$

where Γ is the circulation due to the filament, $\mathbf{t}$ the tangent vector along the filament as given in Eq. (61), and J the Jacobian determinant of the transformation from (x,y,z) to (a,b,c),

$$J=\det\frac{\partial(x,y,z)}{\partial(a,b,c)}. \quad (B3)$$

This expression for the vorticity can be seen to be correct, first because it vanishes off the filament, second it has the correct direction, and third it yields the proper circulation. Indeed, consider a surface S that intersects the filament once, then we may parametrize this surface by the (b,c) coordinates (provided S is small enough in extent) since the surfaces of constant a are transverse to the filament by assumption. We have

$$\int_S\boldsymbol{\omega}\cdot d\mathbf{A}=\Gamma\int db\,dc\,J^{-1}\mathbf{t}\cdot(\mathbf{e}_b\times\mathbf{e}_c)\times\delta(b-B(a))\delta(c-C(a))=\Gamma, \quad (B4)$$

since $\mathbf{t}\cdot(\mathbf{e}_b\times\mathbf{e}_c)=\mathbf{e}_a\cdot(\mathbf{e}_b\times\mathbf{e}_c)=J$. Now we insert our expression for the vorticity into the formula (B1) for the symplectic two form. Whereupon the change of variables

$(x, y, z) \to (a, b, c)$ with $d^3x = J da\, db\, dc$, and δ function integrals over b and c yield,

$$\Omega(\mathbf{n}, \mathbf{m}) = \Gamma \int da\, \mathbf{t} \cdot (\mathbf{n} \times \mathbf{m}). \tag{B5}$$

It is now useful to expand $\mathbf{n}$ and $\mathbf{m}$ in terms our orthogonal triad,

$$\begin{aligned} \mathbf{n}(a) &= n^1(a)\mathbf{e}_a + n^2(a)\mathbf{e}_b + n^3(a)\mathbf{e}_c, \\ \mathbf{m}(a) &= m^1(a)\mathbf{e}_a + m^2(a)\mathbf{e}_b + m^3(a)\mathbf{e}_c, \end{aligned} \tag{B6}$$

where orthogonality of $\mathbf{n}$ and $\mathbf{m}$ to $\mathbf{t}$ demands that

$$m^1 = -\frac{m^2 B' |\mathbf{e}_b|^2 + m^3 C' |\mathbf{e}_c|^2}{|\mathbf{e}_a|^2}, \tag{B7}$$

with a similar expression for n^1. So we may think of the coefficients $m^2(a)$ and $m^3(a)$ as determining the normal vector field $\mathbf{m}(a)$. Using this, the symplectic two form (B5) can (after some algebra) be written

$$\Omega(\mathbf{n}, \mathbf{m}) = \Gamma \int da\, \frac{\tau |\mathbf{t}|^2}{|\mathbf{e}_a|^2} (n^2 m^3 - m^2 n^3), \tag{B8}$$

where τ is the Jacobian along the filament given in Eq. (38). Now we discuss how Poisson brackets are computed using such a formula. The Poisson bracket is an operation which given a pair of functions (here functionals) on the symplectic manifold, yields a third such function. In terms of the symplectic two form, one has the following formula for the Poisson bracket of functions F and G:

$$\begin{aligned} \{F, G\} &= \Omega(\mathbf{n}_F, \mathbf{n}_G) \\ &= \Gamma \int da\, \frac{\tau |\mathbf{t}|^2}{|\mathbf{e}_a|^2} (n_F^2 n_G^3 - n_G^2 n_F^3), \end{aligned} \tag{B9}$$

where $\mathbf{n}_F$ and $\mathbf{n}_G$ are the *Hamiltonian vector fields* associated with F and G, respectively, which themselves are defined in terms of the symplectic two form. In the more classical language of Hamiltonian mechanics, F would be the Hamiltonian and its associated Hamiltonian vector field would be the vector field $(\partial F/\partial \mathbf{p}, -\partial F/\partial \mathbf{q})$. Of course in our case we do not yet know what the q's and p's are and we must compute $\mathbf{n}_F$ by using the symplectic two form directly. $\mathbf{n}_F$ is a normal vector field on the vortex filament and is obtained in the following way. Let $\mathbf{m}$ be another (arbitrary) normal vector filed defined on the filament; now displace the filament infinitesimally using $\mathbf{m}$. Let $\delta_{\mathbf{m}} F$ be the resulting (infinitesimal) change in F. Then $\mathbf{n}_F$ is the normal vector field on the filament defined by the following requirement:

$$\begin{aligned} \delta_{\mathbf{m}} F &= \Omega(\mathbf{n}_F, \mathbf{m}) \\ &= \Gamma \int da\, \frac{\tau |\mathbf{t}|^2}{|\mathbf{e}_a|^2} (n_F^2 m^3 - m^2 n_F^3), \end{aligned} \tag{B10}$$

valid for arbitrary $\mathbf{m}$. The properties of Ω as a symplectic two form guarantee that $\mathbf{n}_F$ exists and is unique. We are interested in the special cases where F is the local functional $B(a)$ or $C(a)$, and for these we may readily compute the infinitesimal changes as

$$\begin{aligned} \delta_{\mathbf{m}} B(a) &= m^2(a) - B'(a) m^1(a), \\ \delta_{\mathbf{m}} C(a) &= m^3(a) - C'(a) m^1(a). \end{aligned} \tag{B11}$$

These expressions were written down by the following reasoning. Since a is our independent variable, the infinitesimal changes which we have denoted by the symbol $\delta_{\mathbf{m}}$ are by definition taken with a fixed. The total infinitesimal change of the coordinates (a, b, c), which we shall denote using the symbol $\Delta_{\mathbf{m}}$, is by definition

$$\Delta_{\mathbf{m}}(a, b, c) = \mathbf{m}. \tag{B12}$$

For changes in $B(a)$ and $C(a)$ with u fixed we then have

$$\begin{aligned} \delta_{\mathbf{m}} B(a) &= \Delta_{\mathbf{m}} b - B'(a) \Delta_{\mathbf{m}} a, \\ \delta_{\mathbf{m}} C(a) &= \Delta_{\mathbf{m}} c - C'(a) \Delta_{\mathbf{m}} a, \end{aligned} \tag{B13}$$

so the expressions given in Eqs. (B11) are indeed the required variations. Now Eq. (B7) can be used to eliminate m^1 in favor of m^2 and m^3 in Eq. (B11). This gives

$$\begin{aligned} \delta_{\mathbf{m}} B(a) &= \frac{1}{|\mathbf{e}_a|^2} [|\mathbf{e}_c|^2 B' C' m^3 + (|\mathbf{e}_a|^2 + |\mathbf{e}_b|^2 B'^2) m^2], \\ \delta_{\mathbf{m}} C(a) &= \frac{1}{|\mathbf{e}_a|^2} [(|\mathbf{e}_a|^2 + |\mathbf{e}_c|^2 C'^2) m^3 + |\mathbf{e}_b|^2 B' C' m^2]. \end{aligned} \tag{B14}$$

Now writing

$$\begin{aligned} \mathbf{n}_B &= n_B^1 \mathbf{e}_a + n_B^2 \mathbf{e}_b + n_B^3 \mathbf{e}_c, \\ \mathbf{n}_C &= n_C^1 \mathbf{e}_a + n_C^2 \mathbf{e}_b + n_C^3 \mathbf{e}_c, \end{aligned} \tag{B15}$$

and using Eqs. (B10) and (B14), we can read off the coefficients $n_B^2, n_B^3, n_C^2, n_C^2$, needed for the computation of the Poisson bracket as given in Eq. (B9),

$$\begin{aligned} n_{B(a)}^2(a') &= \frac{\delta(a - a')}{\Gamma \tau |\mathbf{t}|^2} |\mathbf{e}_c|^2 B' C', \\ n_{B(a)}^3(a') &= -\frac{\delta(a - a')}{\Gamma \tau |\mathbf{t}|^2} (|\mathbf{e}_a|^2 + |\mathbf{e}_b|^2 B'^2), \\ n_{C(a)}^2(a') &= \frac{\delta(a - a')}{\Gamma \tau |\mathbf{t}|^2} (|\mathbf{e}_a|^2 + |\mathbf{e}_c|^2 C'^2), \\ n_{C(a)}^3(a') &= -\frac{\delta(a - a')}{\Gamma \tau |\mathbf{t}|^2} |\mathbf{e}_b|^2 B' C'. \end{aligned} \tag{B16}$$

This can now be inserted into the formula for the Poisson bracket (B9) to yield (after some algebra) the desired Poisson brackets

$$\begin{aligned} \{B(a), B(a')\} &= \{C(a), C(a')\} = 0, \\ \{B(a), C(a')\} &= \frac{1}{\Gamma \tau(a)} \delta(a - a'). \end{aligned} \tag{B17}$$

APPENDIX C: VARIATION OF THE VORTEX FILAMENT VELOCITY POTENTIAL

In this appendix we will give a proof of the formula (71) for the variation of the velocity potential $(\Phi_f)_s$ of the vortex filament evaluated on the free surface. We will actually give a general formula for $\delta\Phi_f$ for the case when $\mathbf{x}$ is not a point on the filament itself, which will then be valid on the free surface, since by assumption our filament is in the interior of the fluid.

In chapter two of Batchelor's fluid mechanics text [17], one finds the following formula for the velocity potential of the velocity field associated with a vortex filament:

$$\Phi_f(\mathbf{x}) = -\frac{\Gamma}{4\pi}\int \mathbf{q}(\mathbf{x}-\mathbf{x}')\cdot d\mathbf{S}(\mathbf{x}'), \tag{C1}$$

where

$$\mathbf{q}(\mathbf{x}-\mathbf{x}') = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{C2}$$

and the integral in $\mathbf{x}'$ is over any surface whose boundary is the filament. If the filament extends to infinity, then it must be "closed at infinity" in a suitable manner to define the surface for the integration. Since this formula for Φ_f involves a surface integral, it is not *a priori* obvious that the variation $\delta\Phi_f$ will depend on the infinitesimal displacement $\delta\mathbf{X}(a)$ of the boundary only. However on physical grounds it is clear that it must be so, at least away from the filament.

Now let (u,v) be parameters for the surface $\mathbf{S}(\mathbf{x}')$, so that $\mathbf{x}'(u,v)$ is a point on the surface. Then by definition of surface integral, Eq. (C1) can be written

$$\Phi_f(\mathbf{x}) = -\frac{\Gamma}{4\pi}\int du\,dv\,\mathbf{q}(\mathbf{x}-\mathbf{x}'(u,v))\cdot\left(\frac{\partial\mathbf{x}'}{\partial u}\times\frac{\partial\mathbf{x}'}{\partial v}\right). \tag{C3}$$

Now it is useful to define

$$\mathbf{A} = \frac{\partial\mathbf{x}'}{\partial u}\times\frac{\partial\mathbf{x}'}{\partial v}, \tag{C4}$$

and

$$\sigma_{jk} = \frac{\partial x'_j}{\partial u}\frac{\partial x'_k}{\partial v} - \frac{\partial x'_j}{\partial v}\frac{\partial x'_k}{\partial u}, \tag{C5}$$

so that

$$\epsilon_{ijk}A_i = \sigma_{jk}, \tag{C6}$$

where ϵ_{ijk}, is the three-dimensional antisymmetric symbol, and summation over repeated indices is understood. We now can turn to computing $\delta\Phi_f(\mathbf{x})$, for Φ_f given by Eq. (C3). At this stage, the symbol δ refers to variations of the surface $\mathbf{x}'(u,v)$. We have

$$\delta\Phi_f(\mathbf{x}) = -\frac{\Gamma}{4\pi}\int du\,dv\left[[(\delta\mathbf{x}'\cdot\boldsymbol{\nabla}')\mathbf{q}]\cdot\mathbf{A} + \mathbf{q}\cdot\left(\frac{\partial\delta\mathbf{x}'}{\partial u}\times\frac{\partial\mathbf{x}'}{\partial v} - \frac{\partial\delta\mathbf{x}'}{\partial v}\times\frac{\partial\mathbf{x}'}{\partial u}\right)\right], \tag{C7}$$

where $\boldsymbol{\nabla}'$ denotes gradient with respect to $\mathbf{x}'$. The first term above is already in the form of a surface integral, so we leave it as it is. The second term can be recognized as a surface integral by using the chain rule, to convert the (u,v) derivatives of $\delta\mathbf{x}'$, to derivatives with respect to $\mathbf{x}'$,

$$\begin{aligned}\int du\,dv\left[\mathbf{q}\cdot\left(\frac{\partial\delta\mathbf{x}'}{\partial u}\times\frac{\partial\mathbf{x}'}{\partial v} - \frac{\partial\delta\mathbf{x}'}{\partial v}\times\frac{\partial\mathbf{x}'}{\partial u}\right)\right] &= \epsilon_{ijk}\int du\,dv\,q_i\frac{\partial\delta x'_j}{\partial x'_n}\sigma_{nk}\\ &= \epsilon_{ijk}\epsilon_{lnk}\int du\,dv\,q_i\frac{\partial\delta x'_j}{\partial x'_n}A_l\\ &= (\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})\int du\,dv\,q_i\frac{\partial\delta x'_j}{\partial x'_n}A_l\\ &= \int du\,dv\,[\mathbf{q}\,(\boldsymbol{\nabla}'\cdot\delta\mathbf{x}') - (\mathbf{q}\cdot\boldsymbol{\nabla}')\delta\mathbf{x}']\cdot\mathbf{A}.\end{aligned}$$

Therefore we have

$$\delta\Phi_f(\mathbf{x}) = \frac{\Gamma}{4\pi}\int[\,(\mathbf{q}\cdot\boldsymbol{\nabla}')\delta\mathbf{x}' - \mathbf{q}\,(\boldsymbol{\nabla}'\cdot\delta\mathbf{x}') - (\delta\mathbf{x}'\cdot\boldsymbol{\nabla}')\mathbf{q}]\cdot d\mathbf{S}(\mathbf{x}'). \tag{C8}$$

Now we shall use the fact that $\mathbf{x}$ is not on the filament, and so the bounding surface $\mathbf{S}(\mathbf{x}')$ can be chosen such that

$$\boldsymbol{\nabla}'\cdot\mathbf{q} = -4\pi\delta(\mathbf{x}-\mathbf{x}') = 0. \tag{C9}$$

Therefore by adding a term $\delta\mathbf{x}'(\nabla'\cdot\mathbf{q})$ to the integrand, we may recognize it as

$$\delta\Phi_f(\mathbf{x}) = \frac{\Gamma}{4\pi}\int\boldsymbol{\nabla}'\times(\delta\mathbf{x}'\times\mathbf{q})\cdot d\mathbf{S}(\mathbf{x}'). \tag{C10}$$

Now Stokes theorem can be applied to express this as a line integral around the filament, giving

$$\delta\Phi_f(\mathbf{x}) = \frac{\Gamma}{4\pi}\int d\mathbf{x}'\cdot(\delta\mathbf{x}'\times\mathbf{q}). \tag{C11}$$

Further, parametrizing this integral using parameter a, we get

$$\delta\Phi_f(\mathbf{x}) = \frac{\Gamma}{4\pi}\int da\,\mathbf{t}(a)\cdot[\delta\mathbf{X}(a)\times\mathbf{q}(\mathbf{x}-\mathbf{X}(a)], \tag{C12}$$

which can be also expressed

$$\delta\Phi_f(\mathbf{x}) = \frac{\Gamma}{4\pi}\int da\,\boldsymbol{\nabla}\cdot\left(\frac{\delta\mathbf{X}(a)\times\mathbf{t}(a)}{|\mathbf{x}-\mathbf{X}(a)|}\right). \tag{C13}$$

This completes the proof of formula (71).

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